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# On minimal vertical singular diffusion preventing overturning (Viscosity Solutions of Differential Equations and Related Topics)

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# On minimal vertical singular diffusion preventing overturning

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## 1 Introduction

This is a preliminary version of our work to a continuation of recent works [9], [10] of the second author.

In [9] we introduce the notion of proper viscosity solutions for a class of equations whose solutions may develop jump discontinuities. The class contains (scalar) conservation laws as special examples and a proper viscosity solution is essentially equivalent to an entropy solution for conservation laws. In [10] we propose to interpret this evolution as a result of the vertical singular diffusion. By a formal argument we have noted in [10] that there is a threshold value of the strength of the vertical diffusion such that it prevents overturning of the solution.

In this paper we give a rigorous proof for the fact that a solution develops overturning if the strength  $M$  of the vertical diffusion is smaller than the critical value by studying the Riemann problem for the Burgers equation:

$$u_t + uu_x = 0, \quad (1.1)$$

$$u(x, 0) = (\operatorname{sgn} x)d/2. \quad (1.2)$$

If one views the graph of  $u$  as a level set of auxiliary function  $\psi(x, y, t)$ ,  $\psi$  must satisfy

$$\psi_t + y\psi_x = 0. \quad (1.3)$$

If we consider (1.3) in  $\mathbf{R}^2 \times (0, T)$ , each level set of  $\psi$  moves by (1.1) if it is represented by the graph of a function  $u = u(x, t)$ . This formulation is successful to track discontinuous solutions for

$$u_t + H(u, u_x) = 0$$

if  $r \mapsto H(r, p)$  is nondecreasing so that solution does not develop discontinuities if the initial data is continuous [12]. However, for (1.1) the zero level set of the solution of (1.3) certainly overturn if initially

$$\{(x, y); \psi(x, y, 0) > 0\} = \{(x, y); y < -d/2\} \cup \{(x, y); x < 0, -d/2 \leq y < d/2\}; \quad (1.4)$$

in fact, the zero level set  $\psi = 0$  for  $t > 0$  cannot be viewed as the graph of a single valued function in any sense.

In [10] we propose to add the vertical diffusion term

$$\psi_t + y\psi_x = M|\nabla\psi|\partial_y(\psi_y/|\psi_y|). \quad (1.5)$$

A formal argument [10, Theorem 2.1] reflecting [3] says that if  $M$  is large so that

$$V_I \geq V - 2M \quad \text{on} \quad I = (-d/2, d/2), \quad (1.6)$$

then the zero level set of  $\psi$  with initial condition (1.4) does not overturn and equals the graph of the entropy solution of (1.1), (1.2). Here  $V(\eta) = -\eta^2/2$  which is the primitive of  $-y$  and  $V_I$  denotes its convex hull in  $I$ . An elementary calculation shows that the minimum value  $M_0$  of  $M$  satisfying (1.6) is  $d^2/16$ . In the numerical simulation [16] we also observe that the overturning occurs if and only if  $M < M_0 = d^2/16$ . (There  $I$  is replaced by  $(a, b)$  but the value of  $M_0$  equals  $(b - a)^2/16$ .)

In this paper we show analytically that  $M_0$  is optimal in the sense that if  $M < M_0$ , the overturning is not prevented. It is also possible to prove that the overturning does not occur  $M \geq M_0$  for more general equations but we shall discuss this problem in one of forthcoming papers.

Although the level set method (see e.g. [8]) allowing the singular diffusivity is well-studied by [4], [5], [6], the equation handled there is spatially homogeneous and excludes (1.5). Instead of developing a general theory for (1.5) we rather study its approximation. In fact, we shall prove that there is a sequence of level set equations

$$\psi_t + y\psi_x = M|\nabla\psi| \operatorname{div}(\nabla\gamma_\epsilon(-\nabla\psi)) \quad (1.7)$$

approximating (1.5) such that the limit of zero level set of  $\psi = \psi^\epsilon$  develops ‘overturning’ if  $M < M_0$ . Here  $\gamma_\epsilon \in C^2(\mathbf{R}^2 \setminus \{0\})$  is convex and positively homogeneous of degree one.

The main idea of the proof is to convert the problem of evolution of  $\{\psi = 0\}$  to the evolution of  $x = v(y, t)$  starting with  $v(y, 0) = 0$ . (For this purpose we assume that  $\nabla^2\gamma(0, 1) = 0$  so that the line segment on the line  $y = \pm d/2$  does not move.) We study the equation for  $v$  derived from (1.7) and prove that it converges to a function which has strictly monotone increasing part in  $y$  if  $M < M_0$ . This means that ‘overturning’ occurs. Unfortunately, the boundary condition for  $v$  at  $y = \pm d/2$  is not conventional. It is formally equals the Neumann condition

$$v_y(\pm d/2, t) = -\infty.$$

This is hard to handle so we estimate from above and below by solutions of a homogeneous Neumann problem and on inhomogeneous Dirichlet problem. We prove that solutions of the latter two problems converges to the same function having desired property.

## 2 Explicit solutions for some inhomogeneous very singular diffusion equations

We consider a singular degenerate parabolic equation for  $v = v(\eta, t)$  of the form

$$v_t = M(\operatorname{sgn} v_\eta)_\eta + \eta \quad \text{in } I \times (0, \infty), \quad (2.1)$$

$$v = 0 \quad \text{on } \partial I \times (0, \infty), \quad (2.2)$$

$$v|_{t=0} = 0 \quad (2.3)$$

with  $I = (-d/2, d/2)$ , where  $M > 0$  is a parameter. Since  $(\operatorname{sgn} v_\eta)_\eta$  formally equals  $\delta(v_\eta)v_{\eta\eta}$ , the diffusion is degenerate for  $v_\eta \neq 0$  and is very strong for  $v_\eta = 0$ . Naively, the meaning of a ‘solution’ is not clear. Fortunately, the theory of nonlinear semigroups [15] or subdifferential equations provides a suitable notion of a solution. We shall briefly review its notion and give an explicit representation formula of the solution.

We first give a subdifferential interpretation of the problem (2.1)-(2.3). For  $v \in H = L^2(I)$  we associate the energy  $E(v)$  defined by

$$E(v) := \int_{\mathbf{R}} \{M|\tilde{v}_\eta(\eta)| - \eta\tilde{v}(\eta)\} d\eta \quad \text{if } v \in BV(I)$$

and  $E(v) := \infty$  if  $v \notin BV(I)$ . Here  $BV(I)$  denotes the space of functions with bounded variation in  $I$  and  $\tilde{v}$  denotes the extension of  $v$  to  $\mathbf{R}$  such that  $\tilde{v} = 0$  outside  $I$ . The integral  $\int_{\mathbf{R}} |\nabla \tilde{v}(n)| d\eta$  denotes the total variation of  $\tilde{v}$  in  $\mathbf{R}$ . Then as in [7, the first lemma in §2] the functional  $E$  is convex, lower semicontinuous in the Hilbert space  $H$  equipped with the standard inner product  $(f, g) = \int_I f g d\eta$ . Note that (2.1) is formally a gradient flow of  $E$ . Thus we formulate the problem (2.1)-(2.3) as

$$\frac{dv}{dt} \in -\partial E(v), \quad (2.4)$$

$$v(0) = 0, \quad (2.5)$$

where  $\partial E$  denotes the subdifferential of  $E$  in  $H$ . A general theory [15], [1] yields that there is a unique solution  $v$  of (2.4) and (2.5) in the sense that

- (i)  $v \in C([0, \infty), H)$  i.e.,  $v$  is continuous from the time interval  $[0, \infty)$  to  $H$ . Moreover,  $v$  satisfies (2.5).

- (ii)  $v$  is absolutely continuous with values in  $H$  on each compact set in  $(0, \infty)$  and solves (2.4) for almost all  $t \geq 0$ .

As well-known (e.g. [1], see also [7, §2]) the solution  $v(t)$  is right-differentiable at all  $t > 0$  with values in  $H$  and its right derivative  $d^+v/dt$  satisfies

$$\frac{d^+v}{dt} = -\partial^0 E(v) \quad \text{for all } t > 0. \quad (2.6)$$

where  $\partial^0 E(v)$  is the canonical restriction (or minimal section) of  $\partial E(v)$ , i.e.,  $\partial^0 E(v)$  is the unique element of the closed convex set  $\partial E(v)$  which is closest to the origin of  $H$ . Moreover, we have another definition of solution equivalent to (i) (ii). Namely,  $v$  is the solution of (2.4) and (2.5) if and only if  $v$  fulfills (i) and

- (ii)'  $v$  is absolutely continuous with values in  $H$  on each compact set in  $(0, \infty)$  and solves (2.6) for all  $t > 0$ .

Here and hereafter by solution of (2.1)-(2.3) we mean that  $v$  satisfies (i) and (ii)'. Fortunately, the solution can be represented in an explicit formula.

**Lemma 2.1.** *Let  $v$  be the solution of (2.1)-(2.3). Then  $v$  is represented by*

$$v(\eta, t) = tv_1(\eta), \quad t \geq 0 \quad (2.7)$$

with  $v_1$  satisfying

$$\begin{aligned} v_1(\eta) &= \min(\eta, (\frac{d}{2} - 2M^{1/2})_+) & \text{for } \eta \in [0, \frac{d}{2}) \\ v_1(\eta) &= -v_1(-\eta) & \text{for } \eta \in (-\frac{d}{2}, 0], \end{aligned}$$

where  $\alpha_+ = \max(\alpha, 0)$ . In particular,  $v_1 \equiv 0$  if and only if  $M \geq d^2/16$  and otherwise  $v_1$  has a strictly increasing part.

**Remark 2.2.** (i) If we replace the homogeneous Dirichlet condition (2.2) by the homogeneous Neumann condition

$$v_\eta = 0 \quad \text{on } \partial I \times (0, T), \quad (2.2')$$

the solution of (2.1) with (2.2'), (2.3) is the same as in (2.7). Here we should replace the definition of  $E$  by

$$E_N(v) := \int_I \{M|v_\eta| - \eta v\} d\eta \quad \text{if } v \in BV(I) \quad (2.8)$$

and  $E_N(v) := \infty$  if  $v \notin BV(I)$  so that (2.1), (2.2') (2.3) is formulated by (2.4), (2.5) with  $E$  replaced by  $E_N$ .

(ii) We may replace the homogeneous Dirichlet condition (2.2) by inhomogeneous Dirichlet condition

$$v = \mp R \quad \text{at} \quad \eta = \pm d/2. \quad (2.2'')$$

The solution of (2.1) with (2.2''), (2.3) is the same as in (2.7) for  $R > 0$ . Here we should replace  $E$  by

$$E_R(v) := \int_{\mathbf{R}} \{M|\bar{v}_\eta| - \eta\bar{v}\} d\eta \quad \text{if} \quad v \in BV(I) \quad (2.9)$$

and  $E_R(v) := \infty$  if  $v \notin BV(I)$ . The extension  $\bar{v}$  of  $v$  equals  $-R$  for  $\eta \geq d/2$  and  $R$  for  $\eta \leq -d/2$ . The equation (2.1), (2.2''), (2.3) is now formulated by (2.4), (2.5) with  $E$  replaced by  $E_R$ .

To show these statements it suffices to verify (2.6) as in [3].

### 3 Neumann problems for some non-uniform parabolic equations

To study solutions of problems approximating (2.1)-(2.3) we consider the Neumann problem:

$$v_t = a(v_\eta)v_{\eta\eta} + \eta \quad \text{in} \quad I \times (0, \infty), \quad (3.1)$$

$$v_\eta = -\alpha \quad \text{on} \quad \partial I \times (0, \infty), \quad (3.2)$$

$$v|_{t=0} = 0. \quad (3.3)$$

Here  $a \in C^1(\mathbf{R})$  is assumed to be positive and  $\alpha$  is a non-negative constant. Since  $v_\eta$  of (3.1) solves

$$v_{\eta t} = (a(v_\eta)v_{\eta\eta})_\eta + 1, \quad (3.4)$$

by the maximum principle we have an a priori bound  $|v_\eta(n, t)| \leq \max(t, \alpha)$  for  $v_\eta$ . So in  $I \times (0, T)$  with  $T > 0$  we may assume that equation is uniformly parabolic by restricting  $a$  on  $[-\max(T, \alpha), \max(T, \alpha)]$ . A general theory of parabolic equations [14] yields an unique global classical solution  $v \in C^{2,1}(I \times [0, \infty)) \cap C^{2,1}(\bar{I} \times (0, \infty))$  of (3.1)-(3.3).

Our main goal in this section is to prove several properties of the solution of (3.1)-(3.3).

**Theorem 3.1.** *Let  $v^\alpha$  be the solution of (3.1)-(3.3) with  $\alpha \geq 0$ .*

(i) (Symmetry).  $v^\alpha(\eta, t) = -v^\alpha(-\eta, t)$  for  $\eta \in I$ ,  $t \geq 0$ . In particular,  $v^\alpha(0, t) = 0$  for  $t > 0$ .

(ii) (Concavity).  $v^\alpha(\eta, t) \leq \eta t$ ,  $v_t^\alpha(\eta, t) \leq \eta$  for  $\eta \in I_+$ ,  $t \geq 0$  with  $I_+ = (0, d/2)$ . In particular,  $v_{\eta\eta}^\alpha \leq 0$  in  $I_+ \times (0, \infty)$ .

(iii) (Monotonicity).  $v^\alpha \leq v^\beta$  in  $I_+ \times (0, \infty)$  if  $\alpha \geq \beta \geq 0$ . Moreover  $v_\eta^\alpha \leq v_\eta^\beta$  in  $I_+ \times (0, \infty)$  if  $\alpha \geq \beta \geq 0$ .

(iv) (Lower bound). Assume that

$$c_0 := \int_{-\infty}^0 a(\tau) d\tau \leq \frac{d^2}{8} \quad (3.5)$$

and

$$c_1 := \int_{-\infty}^0 |\tau| a(\tau) d\tau < \infty. \quad (3.6)$$

Then  $v^\alpha(\eta, t) \geq -c_0 c_1$  for  $\eta \in [0, d/2], t \geq 0$ .

*Proof.* (i) Since  $-v^\alpha(-\eta, t)$  solves (3.1)-(3.3), the uniqueness of a solution yields the symmetry.

(ii) Clearly  $\eta t$  is a supersolution of (3.1)-(3.3) in  $I_+ \times (0, \infty)$  with zero boundary condition at  $\eta = 0$  so the comparison principle yields  $v \leq \eta t$  in  $I_+ \times (0, \infty)$ . We differentiate (3.1), (3.2) in  $t$  to get

$$\begin{aligned} w_t &= a(v_\eta^\alpha) w_{\eta\eta} + a'(v_\eta^\alpha) w_\eta v_{\eta\eta}^\alpha \quad \text{in } I \times (0, \infty) \\ w_\eta(d/2, t) &= 0, \quad w(0, t) = 0 \quad (\text{by (i)}) \end{aligned}$$

for  $w = v_t^\alpha$ . Since  $v_t^\alpha \leq \eta$  at  $t = 0$  on  $I_+$  by  $v^\alpha \leq \eta t$ , the maximum principle implies that  $w \leq \eta$  in  $[0, d/2] \times [0, \infty)$ . The concavity follows from  $v_t \leq \eta$  and the equation (3.1) since  $a > 0$ .

(iii) For  $\beta \leq \alpha$  the solution  $v^\beta$  is a supersolution of (3.1)-(3.3) with  $v = 0$  at  $\eta = 0$  in  $I_+ \times (0, \infty)$ , the comparison principle yields  $v^\alpha \leq v^\beta$  in  $I_+ \times (0, \infty)$ . Since  $v^\alpha \leq v^\beta$  and  $v^\alpha = v^\beta = 0$  at  $\eta = 0$ , we observe that  $v_\eta^\alpha \leq v_\eta^\beta$  at  $\eta = 0$ . Since  $v_\eta^\beta$  solves (3.4) and  $v_\eta^\alpha \leq v_\eta^\beta$  at  $\eta = d/2$ , the comparison principle yields  $v_\eta^\alpha \leq v_\eta^\beta$  in  $I_+ \times (0, \infty)$ .

(iv) As in the next Lemma we shall construct a time independent subsolution  $f = f_\alpha$  for (3.1)-(3.3) in  $I_+ \times (0, \infty)$  with the zero-boundary condition at  $\eta = 0$  such that  $f_\alpha \geq -c_0 c_1$ . Once such a subsolution is constructed, the comparison principle yields the bound  $v^\alpha \geq -c_0 c_1$  for  $v^\alpha$ .

**Lemma 3.2.** Assume that (3.5) holds. Then there exists a unique  $\sigma \in I_+ = (0, d/2)$  and a  $C^1$  function  $f = f_\alpha$  on  $\tilde{I}_+$  such that

$$-(A(f'(\eta)))' = \eta \quad \text{on } I_+, \quad (3.7)$$

$$f'(d/2) = -\alpha, \quad f'(\sigma) = f(\sigma) = 0, \quad (3.8)$$

where  $A(q) = \int_0^q a(\tau) d\tau$  and  $f'$  denotes the derivative of  $f$ . If moreover  $a$  satisfies (3.6), then

$$-c_0 c_1 \leq \inf\{f_\alpha(\eta); \quad \eta \in [0, d/2], \alpha \geq 0\} = \inf\{f_\alpha(d/2); \alpha \geq 0\} \quad (3.9)$$

(The zero-extension of  $f_\alpha$  to  $[0, \sigma]$  is still denoted by  $f_\alpha$ ).

*Proof.* Integrating (3.7) from  $\sigma$  to  $\eta$  yields

$$-A(f'(\eta)) = (\eta^2 - \sigma^2)/2 \quad (3.10)$$

since  $f'(\sigma) = 0$ . Since  $A(p) \leq d^2/8$  for  $p \leq 0$  by (3.5), there is unique  $\sigma \in I_+$  such that

$$-A(-\alpha) = \frac{1}{2} \left( \frac{d}{2} \right)^2 - \frac{\sigma^2}{2}.$$

We fix such a  $\sigma$  and then taking the inverse  $A^{-1}$  of (3.10) to get

$$f'(\eta) = A^{-1}((\sigma^2 - \eta^2)/2), \quad \eta \in [\sigma, d/2]. \quad (3.11)$$

Integrating this with  $f(\sigma) = 0$  we obtain the solution  $f$  and  $\sigma \in I_+$  satisfying (3.7), (3.8).

By (3.11)  $f'(\eta) \leq 0$  in  $I_+$  so  $\inf_{I_+} f = f(d/2)$ . Thus to prove (3.9) it suffices to prove that

$$\inf_{\alpha} f_\alpha(d/2) > -\infty. \quad (3.12)$$

Integrating (3.11) over  $[\sigma, d/2]$  to get

$$\begin{aligned} -f_\alpha(d/2) &= -\int_{\sigma}^{d/2} A^{-1}((\sigma^2 - \eta^2)/2) d\eta \\ &= -\int_{A(-\alpha)}^0 A^{-1}(\xi) \xi d\xi \leq -A(-\infty) \int_{A(-\infty)}^0 A^{-1}(\xi) d\xi. \end{aligned}$$

Since

$$-\int_{A(-\infty)}^0 A^{-1}(\tau) d\tau = \int_{-\infty}^0 (A(p) - A(-\infty)) dp = \int_{-\infty}^0 |p| a(p) dp = C_0$$

we now obtain that  $-f_\alpha(d/2) \leq C_0$ .  $\square$

## 4 Approximate problems

Let  $v^\alpha$  be the solution of (3.1)-(3.3). We define  $v^\infty$  by

$$v^\infty(\eta, t) = \inf_{\alpha > 0} v^\alpha(\eta, t), \quad \eta \in I_+ = (0, d/2)$$

$$v^\infty(\eta, t) = -v^\infty(-\eta, t), \quad \eta \in (-d/2, 0)$$

$$v^\infty(0, t) = 0.$$

By the monotone properties and bounds (Theorem 3.1)  $v^\infty$  is well-defined and  $\eta \mapsto v^\infty(\eta, t)$  is  $C^1$  and concave in  $I_+$ .

Our goal in this section is to prove the convergence of  $v^\infty$  to  $v$  in (2.7) when  $f^q a$  approximates  $M \operatorname{sgn} q$ .



**Theorem 4.1.** Assume that  $a = a^\varepsilon \in C^1(\mathbf{R})$ ,  $a^\varepsilon > 0$  satisfies (3.5) and (3.6). Assume that  $c_0^\varepsilon$ ,  $c_1^\varepsilon$  defined by (3.5), (3.6) with  $a = a^\varepsilon$  are bounded as  $\varepsilon \rightarrow 0$ . Assume that  $A^\varepsilon(q) = \int_0^q a^\varepsilon(\tau) d\tau$  converges to  $M \operatorname{sgn} \eta + c$  with some constant  $c$  as  $\varepsilon \rightarrow 0$  (in the sense of monotone graphs). Let  $v_\varepsilon^\infty$  be the solution of (3.1), (3.2), (3.3) with  $a = a^\varepsilon$  and let  $v_\varepsilon^\infty = \inf_{\alpha > 0} v_\varepsilon^\alpha$ . Let  $v$  be the function defined in (2.7). Then  $v_\varepsilon^\infty$  converges to  $v$  as  $\varepsilon \rightarrow 0$  uniformly in every compact subset of  $I \times [0, \infty)$ .

We shall prove this result by estimating  $v_\varepsilon^\infty$  from above by the solution of the homogeneous Neumann problem and from below by that of a nonhomogeneous Dirichlet problem.

## 4.1 Convergence of the Neumann problem

**Proposition 4.2.** Assume that  $A^\varepsilon(q) = \int_0^q a^\varepsilon(\tau) d\tau$  convergence to  $M \operatorname{sgn} \eta + c$  with some constant  $c$  as  $\varepsilon \rightarrow 0$ , where  $a^\varepsilon \in C^1(\mathbf{R})$  and  $a^\varepsilon > 0$ . Let  $v_\varepsilon^0$  be the solution of (3.1)-(3.3) with  $\alpha = 0$ . Then  $v_\varepsilon^0$  converges to  $v$  (defined by (2.7)) as  $\varepsilon \rightarrow 0$  uniformly in  $\bar{I} \times [0, T]$  for any  $T > 0$ .

*Proof.* We formulate the problem (3.1)-(3.3) by using a subdifferential equation  $u_t \in -\partial E_N^\varepsilon(u)$ ,  $u|_{t=0} = 0$ . By a stability theorem of J. Watanabe [17] based on [2] the solution  $v_\varepsilon^0$  converges to a solution  $u$  of  $u_t \in -\partial E_N$  in  $C([0, T], L^2(I))$  for any  $T > 0$ . Since the solution of  $u_t \in -\partial E_N$  with  $u|_{t=0} = 0$  equals  $v$  of (2.7) as in Remark 2.2,  $v_\varepsilon^0 \rightarrow v$  in  $C([0, T], L^2(I))$ . By Theorem 3.1  $v_\varepsilon^0(\eta, t)$  is concave in  $\eta \in I_+$  and  $v_\varepsilon^0 \leq 1$  at  $\eta = 0$ . Since  $v_{\varepsilon\eta}^0(d/2, t) = 0$ , we see that  $v_{\varepsilon_j}^0(\cdot, t_j)$  always contains a uniform convergent subsequence on  $I$  as  $j \rightarrow \infty$  if  $\varepsilon_j \rightarrow 0$ ,  $t_j \in [0, T]$ . Since  $v_\varepsilon^0 \rightarrow v$  in  $C([0, T], L^2(I))$  this implies the uniform convergence of  $v_\varepsilon^0$  in  $\bar{I} \times [0, T]$  as stated in the next lemma whose proof is elementary.

**Lemma 4.3.** Assume that  $u^\varepsilon \rightarrow u$  in  $C([0, T], L^2(\Omega))$  as  $\varepsilon \rightarrow 0$ , where  $\Omega$  is an open set in  $\mathbf{R}^d$ . Assume that  $\{u^{\varepsilon_j}(\cdot, t_j)\}$  has a uniform convergent subsequence in  $\bar{\Omega}$  provided that  $\varepsilon_j \rightarrow 0$ ,  $t_j \in [0, T]$ . Then  $u^\varepsilon \rightarrow u$  uniformly in  $[0, T] \times \bar{\Omega}$ .

## 4.2 Dirichlet problem

We consider the Dirichlet problem for (3.1), (3.3) with  $a = a^\varepsilon$  with the boundary condition

$$v(\pm d/2, t) = \mp R, \quad (4.1)$$

where  $R$  is a positive constant. Let  $v_{R^\varepsilon}$  be the solution of (3.1), (3.3) with (4.1). The solution may not be satisfies (4.1). It can be understood as the limit of a uniformly parabolic problem which approximates (3.1), (3.3) and (4.1). Since we may assume that we conclude that  $v_{R^\varepsilon, \eta\eta} \leq 0$  in  $I_t \times (0, \infty)$ .

**Proposition 4.4** Assume the same hypotheess of Proposition 4.2 concerning  $a^\varepsilon$ . Let  $v_{R^\varepsilon}$  be the solution of (3.1), (3.3) and (4.1). with  $a = a^\varepsilon$ . Then  $v_{R^\varepsilon} \rightarrow v$  as  $\varepsilon \rightarrow 0$  uniformly in each compact subset of  $I \times [0, \infty)$ , where  $v$  is defined by (2.7).

*Proof.* As in the proof of Proposition 4.2 we observe that  $v_{R^\varepsilon} \rightarrow v$  in  $C([0, T], L^2(I))$ . Again  $v_{R^\varepsilon}$  is concave in  $\eta \in I_+$  and  $v_{R^\varepsilon, \eta}(0, t) \leq 1$ . However, there is no control on  $v_{R^\varepsilon, \eta}(d/2, t)$ . All we expect is that  $v_{R^\varepsilon}$  is bounded in  $I_+ \times [0, T]$  and  $v_{R^\varepsilon}$  is concave in  $\eta$ . From these facts we are able to prove that  $v_{R^\varepsilon}(\cdot, t_j)$  has a uniform convergent subsequence in  $[0, d/2 - \delta]$  for each  $\delta > 0$  if  $t_j \in [0, T]$  and  $\varepsilon_j \rightarrow 0$ . By Lemma 4.3 we now conclude that  $v_{R^\varepsilon} \rightarrow v$  in each compact subset of  $I \times [0, \infty)$

*Proof of Theorem 4.1.* By Theorem 3.1 (iii) we see that  $v_\varepsilon^\infty \leq v_\varepsilon^0$  in  $I_+ \times (0, \infty)$ . We take  $R \geq c_0^\varepsilon c_1^\varepsilon$  for small  $\varepsilon > 0$ . Then by the comparison for the Dirichlet problem

$$v_{R^\varepsilon} \leq v_\varepsilon^\alpha \quad \text{in } I_+ \times (0, \infty).$$

since  $v_{R^\varepsilon} = v_\varepsilon^\alpha = 0$  at  $\eta = 0$ . This implies

$$v_{R^\varepsilon} \leq v_\varepsilon^\infty \quad \text{in } I_+ \times (0, \infty).$$

The convergence results (Propositions 4.2, 4.4) yield the convergence  $v_\varepsilon^\infty \rightarrow v$ .  $\square$

## 5 Level set solutions

We consider the level set equation of the form

$$\psi_t + y\psi_x = M|\nabla\psi|\text{div}\{\nabla\gamma(-\nabla\psi/|\nabla\psi|)\} \quad \text{in } \mathbf{R}^2 \times (0, \infty) \quad (5.1)$$

Here  $\gamma$  is a convex, positively homogeneous of degree one in  $\mathbf{R}^2$ . If  $M = 0$ , the set  $\{\psi = 0\}$  formally represents the graph of a solution of the Burgers equation for  $u = u(x, t)$ :

$$u_t + uu_x = 0.$$

We shall use the convention that  $\psi > 0$  below the graph of  $u$ . By a standard theory of the level set equation for each  $\psi_0 \in \text{BUC}(\mathbf{R}^2)$  there is a unique viscosity solution  $\psi \in \text{BUC}(\mathbf{R}^2 \times [0, T])$  for any  $T > 0$  of (5.1) satisfying  $\psi(x, y, t) = \psi_0(x, \eta)$  provided that  $\gamma \in C^2(\mathbf{R} \setminus \{0\})$ ; see [11], [13]. We consider the initial data  $\psi_0$  satisfying

$$\{\psi_0 > 0\} = \{(x, \eta); y < -d/2\} \cup \{(x, \eta); x > 0, y < d/2\} =: D_0.$$

and call the set  $D = \{\psi > 0\}$  is the level set solution (of (5.1)) with the initial data  $D_0$ . The set  $D$  is independent of the choice of  $\psi_0$  and is uniquely determined by  $D_0$ .

Our main goal is to show that if  $M < d^2/16$ , then for a large class of  $\gamma$  such that  $\nabla\gamma(-\nabla\psi/|\nabla\psi|)$  approximating  $\psi_y/|\psi_y|$ , the limit of  $D$  develop ‘overturning’.

**Lemma 5.1.** *Let  $\gamma \in C^2(\mathbf{R}^2 \setminus \{0\})$  be convex and positively homogeneous of degree one. Then*

$$\nabla^2\gamma(0,1) = 0$$

*if and only if  $|q|^3W''(q) \rightarrow 0$  as  $q \rightarrow -\infty$  for  $W(q) = \gamma(1, -q)$ .*

*Proof.* By definition

$$\gamma_2(1, -q) = -W'(q) \quad \text{and} \quad \gamma_{22}(1, -q) = W''(q),$$

where  $\gamma_i = \partial\gamma/\partial p_i$ ,  $\gamma_{ij} = \partial^2\gamma/\partial p_i\partial p_j$ . Since  $\gamma_i$  is positively homogeneous of degree one, we have

$$\gamma_{12}(1, -q) - q\gamma_{22}(1, -q) = 0$$

$$\gamma_{11}(1, -q) - q\gamma_{12}(1, -q) = 0.$$

Thus

$$\gamma_{11}(1, -q) = q^2W''(q), \quad \gamma_{12}(1, -q) = qW'(q).$$

Since  $\gamma_{ij}$  is positively homogeneous of degree  $-1$ ,

$$\gamma_{ij}(1/(1+q^2)^{1/2}, -q/(1+q^2)^{1/2}) = (1+q^2)^{1/2}\gamma_{ij}(1, -q) \rightarrow \gamma_{ij}(0,1)$$

as  $q \rightarrow -\infty$ . Thus  $q^3W''(q) \rightarrow 0$  as  $q \rightarrow \infty$  is equivalent to  $\gamma_{ij}(0,1) = 0$  for all  $1 \leq i, j \leq 2$ .  $\square$

The next lemma relates the level set solution  $D$  and a solution of (3.1), (3.3).

**Lemma 5.2** *Let  $\gamma \in C^2(\mathbf{R} \setminus \{0\})$  be convex and positively homogeneous of degree. Assume that  $|q^3|W''(q) \rightarrow 0$  as  $q \rightarrow -\infty$  for  $W(q) = \gamma(1, -q)$ . Assume that  $W''(q) > 0$ . For  $a(q) = M(1+q^2)^{1/2}W''(q)$  let  $v^\alpha$  the solution of (3.1)-(3.3) and  $v^\infty = \inf_{\alpha>0} v^\alpha$ . Let  $D$  be the level set solution with initial data  $D_0$ . Then*

$$D = \{(x, y, t); y < -d/2\} \cup \{(x, y, t); x < v^\infty(y, t), -d/2 \leq y < d/2\}. \quad (5.2)$$

The proof is not short. We here indicate the idea of the proof.

Step1. The right hand side (denoted  $\tilde{D}$ ) of (5.2) is a solution of (5.1) in the sense that the characteristic function of  $\tilde{D}$  solves (5.1) in the viscosity sense. We use the fact that the straight part of  $\partial\tilde{D} \subset \{y = \pm d/2\}$  does not move because of Lemma 5.1. We also note that  $v_\eta^\infty(\eta, t) \rightarrow -\infty$  as  $\eta \uparrow d/2$ . This is important to prove that  $\tilde{D}$  is the solution of (5.1). Note that if the boundary of  $\tilde{D}$  is written as  $x = v(y, t)$ , then  $v$  satisfies (3.1).

Step.2 The set  $\tilde{D}$  is the level set solution. This can be proved by showing that there is no fattening for  $\tilde{D}$ .

As an application of Theorem 4.1 we have a convergence result.

**Theorem 5.3.** *Let  $\gamma^\varepsilon$  fulfill the assumption of  $\gamma$  in Lemma 5.2 with  $W^\varepsilon(q) = \gamma^\varepsilon(1, -q)$ . Assume that  $W^{\varepsilon'}(q) \rightarrow \operatorname{sgn} q + c$  with some constant  $c$  as  $\varepsilon \rightarrow 0$  in the sense of monotone graphs. Let  $D^\varepsilon$  be the level set solution of (5.1) with  $\gamma = \gamma^\varepsilon$  starting with  $D_0$ . Assume that there is  $r > 0$  such that*

$$\int_{-\infty}^0 (1 + q^2)^{1/2} W^{\varepsilon''}(q) dq \leq r \quad \text{for small } \varepsilon$$

and

$$\sup_{0 < \varepsilon < 1} \int_{-\infty}^0 |q|(1 + q^2)^{1/2} W^{\varepsilon''}(q) dq < \infty.$$

Then  $\bar{D}^\varepsilon$  converges to

$$E = \{(x, y, t); y < -d/2\} \cup \{(x, y, t); x < v(y, t), -d/2 \leq y < d/2\}$$

in the sense of Hausdorff distance topology provided that  $Mr \leq d^2/8$ .

**Example.** If  $W^\varepsilon(q) = \int_0^q \tanh(\tau/\varepsilon) d\tau$ , then

$$\int_{-\infty}^0 (1 + q^2)^{1/2} W^{\varepsilon''}(q) dq \rightarrow 1,$$

so for each  $\delta > 0$ , there is  $\varepsilon_0 > 0$  such that

$$\int_{-\infty}^0 (1 + q^2)^{1/2} W^{\varepsilon''}(q) dq \leq 1 + \delta \quad \text{for } \varepsilon \in (0, \varepsilon_0).$$

The condition

$$\sup_{0 < \varepsilon < 1} \int_{-\infty}^0 q(1 + q^2)^{1/2} W^{\varepsilon''}(q) dq < \infty$$

is evidently fulfilled. Thus the convergence results holds for  $M(1 + \delta) \leq d^2/8$ . If  $\delta > 0$  is taken small so that  $(1 + \delta)/16 < 8$ , then we have a threshold value  $M = d^2/16$  such that if  $M < d^2/16$ , then  $E$  experiences ‘overturning’ in the sense that there is a point  $(x_0, y_0, t_0)$  and  $(x_0, y_1, t_0)$  satisfying  $y_1 < y_0$  such that

$$(x_0, y_0, t_0) \in E \quad \text{while} \quad (x_0, y_1, t_0) \notin E.$$

If  $M \geq d^2/16$ ,  $E = D_0 \times (0, \infty)$  so no overturn occurs.

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